

Unbased rational homotopy theory: a Lie algebra approach

James Maunder

Abstract

In this paper an algebraic model for unbased rational homotopy theory from the perspective of curved Lie algebras is constructed. As part of this construction a model structure for the category of pseudo-compact curved Lie algebras with curved morphisms will be introduced; one which is Quillen equivalent to a certain model structure of unital commutative differential graded algebras, thus extending the known Quillen equivalence of augmented algebras and differential graded Lie algebras.

Contents

Introduction	1
Notation and conventions	3
1. The category of curved Lie algebras	5
2. Analogues of the Chevalley-Eilenberg and Harrison complexes	8
3. Model category of curved Lie algebras.	10
4. Unbased rational homotopy theory	19
References	24

Introduction

Rational homotopy theory of connected spaces was developed by Quillen [Qui69] from the viewpoint of differential graded Lie algebras and by Sullivan [Sul77] from the viewpoint of commutative differential graded algebras. A standard reference for the correspondence between rational connected spaces—in both the pointed and unpointed cases—and commutative differential graded algebras is [BG76]. Section 4 relies heavily upon results of this paper. Within [BG76] a closed model category structure is constructed for the category of non-negatively graded unital commutative differential graded algebras, and further there is shown to be a pair of Quillen adjoint functors between the category of non-negatively graded commutative differential graded algebras and the category of simplicial sets. These Quillen adjoint functors restrict to an equivalence on the homotopy categories of connected commutative differential graded algebras of finite type over \mathbb{Q} , and connected rational nilpotent simplicial sets of finite type over \mathbb{Q} . In [LM15], this correspondence between non-negatively graded unital commutative differential graded algebras and simplicial sets is generalised to create a disconnected rational homotopy theory by extending to the category of \mathbb{Z} -graded unital commutative differential graded algebras. Removing the restriction of non-negative grading, despite seeming relatively harmless, has profound consequences; for example, a commutative algebra concentrated in degree zero is necessarily cofibrant in the category of Bousfield and Gugenheim, but this is not always the case in the category of Lazarev and Markl. The latter category does, however, appear to be more natural as one can, for example, define Harrison-André-Quillen cohomology within it, c.f. [BL05]. The theory of [LM15] relies upon a Quillen adjoint pair of functors between the categories of \mathbb{Z} -graded commutative differential graded algebras and simplicial sets that gives rise to an adjunction on the level of homotopy categories, which restricts to an equivalence between the homotopy category of simplicial sets having a finite number of connected components, each being rational, nilpotent, and of finite type (over \mathbb{Q}) and a certain subcategory of the homotopy category of \mathbb{Z} -graded commutative differential graded algebras (given explicitly loc. cit.). Sections 3 and 4 rely on some of the work developed by Lazarev and Markl in their paper. Further, Lazarev

and Markl constructed a second version of this disconnected rational homotopy theory: one formed using differential graded Lie algebras. This second version was performed by relating the homotopy theory of unital commutative differential graded algebras and differential graded Lie algebras. Relationships of this kind are often referred to as a Koszul duality, the theory of which was established by work of Quillen [Qui69]. Quillen showed there existed a duality between differential graded counital cocommutative coalgebras and differential graded Lie algebras under (quite severe) restrictions on the grading of the objects considered. Subsequently, these restrictions were removed by Hinich [Hin01].

The context of this paper is one in which attention is drawn to the categories of commutative differential graded algebras and pseudo-compact curved Lie algebras. A curved Lie algebra is said to be pseudo-compact if it is the inverse limit of an inverse system of finite dimensional nilpotent curved Lie algebras. This inverse limit is taken using strict morphisms, i.e. using morphisms that commute with the differentials and curvature elements. Herein it will be shown that there exists a Quillen equivalence between the categories of commutative differential graded algebras and pseudo-compact curved Lie algebras. This result is proven by a suitable adaptation of Hinich's methods [Hin01] and influenced by the work of Positselski [Pos11]. Consequently an algebraic model of unbased disconnected rational homotopy theory using pseudo-compact curved Lie algebras is obtained. The condition that the curved Lie algebras be pseudo-compact is necessary, for it cannot be removed even when restricted to connected spaces (or simplicial sets).

Numerous papers now discuss the homotopy theory of differential graded coalgebras over different operads. For example Positselski studied coassociative differential graded coalgebras [Pos11]. However, the coalgebras were assumed to be conilpotent *loc. cit.*, and thus the homotopy theory developed therein is not completely general. Furthermore, Positselski worked with curved objects suggesting that when discussing a Koszul duality in more general cases one side of the Quillen equivalence should be a category consisting of curved objects; a hypothesis that is strengthened by results of [CLM14] and this paper.

Section 1 begins by defining and discussing some of the basic properties of curved Lie algebras and their morphisms, before moving on to discuss the category of pseudo-compact curved Lie algebras and filtrations of such objects. These filtrations are particularly important to this paper and the homotopy theory developed herein; the filtrations play an essential role in defining the correct notion of a weak equivalence in the category of pseudo-compact curved Lie algebras, *c.f.* Section 3. Without these filtrations the usual notion of a weak-equivalence, i.e. a quasi-isomorphism, is simply not fine enough.

Section 2 introduces a pair of adjoint functors between the categories of pseudo-compact curved Lie algebras and unital commutative differential graded algebras; these functors are analogues of the Chevalley-Eilenberg and Harrison complexes (see [Wei94] and [Bar68, Har62] respectively) in homological algebra and are influenced by constructions of Positselski [Pos11] in the associative case.

A model structure for the category of pseudo-compact curved Lie algebras is defined in Section 3; one which by the previously defined pair of adjoint functors is Quillen equivalent to the existing model structure of unital commutative differential graded algebras given by Hinich [Hin97]—the main result of the paper. As remarked above, the notion of a filtered quasi-isomorphism using the filtrations defined in Section 1 plays a fundamental role in the homotopy theory of pseudo-compact curved Lie algebras. The proof of this Quillen equivalence relies upon the duality for associative algebras contained within [Pos11], as the functors defined in Section 2 can be ‘embedded’ into the adjunction given in *op. cit.*, this is similar to the method used in [LM15].

In Section 4 this equivalence of homotopy categories is then applied, in a similar manner to Lazarev and Markl [LM15], in the construction of a disconnected rational homotopy theory from the perspective of pseudo-compact curved Lie algebras. This viewpoint results in a couple of corollaries. The first corollary shows that the Maurer-Cartan simplicial set functor commutes with coproducts when restricted to the correct subcategories of curved Lie algebras and simplicial sets; this result is an analogue of one proven in [LM15], but the proof herein is much more simple due to the material developed. The second corollary constructs Lie models for mapping spaces between two simplicial sets each composed of finitely many connected components where each component is rational, nilpotent, and of finite type. More precisely, it constructs the mapping space as the Maurer-Cartan simplicial set of a certain curved Lie algebra.

Notation and conventions

Throughout this paper it is assumed that all commutative and Lie algebras are over a fixed base field, k , of characteristic 0. There are no further assumptions made about the base field until Section 4, when the field will necessarily be assumed to be \mathbb{Q} . All unmarked tensors will be over the base field, unless stated otherwise. Every graded space will be assumed to be \mathbb{Z} -graded, unless stated otherwise: Section 4, for example, will assume some algebras to be non-negatively graded. It will be assumed that all commutative algebras possess a unit, unless stated otherwise, and likewise that all cocommutative coalgebras possess a counit.

The notation $\langle a, b, c, \dots \rangle$ will be understood to mean the k -vector space spanned by the basis vectors a, b, c, \dots . Similarly, the notation $\hat{L}\langle a, b, c, \dots \rangle$ will be understood to mean the completed free graded Lie algebra on generators a, b, c, \dots . That is subspace of $\prod_{i \geq 1} \langle a, b, c, \dots \rangle^{\otimes i}$ spanned by Lie monomials in $\langle a, b, c, \dots \rangle$. Moreover, given a pseudo-compact graded Lie algebra \mathfrak{g} the notation $\mathfrak{g}\langle a, b, c, \dots \rangle$ will be understood to mean the elements a, b, c, \dots are added freely to \mathfrak{g} in a complete fashion.

This paper will often use an assortment of abbreviations. These abbreviations include: dg for differential graded; dglA for differential graded Lie algebra; cdga for commutative differential graded algebra; CMC for closed model category (in the sense of [Qui69], for a review of this material consult [DS95]); LLP for left lifting property; and RLP for right lifting property.

Recall that a MC (Maurer-Cartan) element of a curved Lie algebra, $(\mathfrak{g}, d, \omega)$ in the homological grading is an element $\xi \in \mathfrak{g}$ with $|\xi| = -1$ such that the Maurer-Cartan equation;

$$\omega + d\xi + \frac{1}{2}[\xi, \xi] = 0,$$

is satisfied. Notice that if $\omega = 0$ (i.e. \mathfrak{g} is a differential graded Lie algebra) then the classical MC equation is recovered:

$$d\xi + \frac{1}{2}[\xi, \xi] = 0.$$

All Lie algebras will be given the homological grading with lower indices, whereas commutative algebras will be given the cohomological grading with upper indices. However, there is an important exception to this rule; the tensor product of a homologically graded Lie algebra with the cohomologically graded cdga Ω of the Sullivan-de Rham forms. In this context, Ω is considered as homologically graded using the relations $\Omega_i := \Omega^{-i}$ for each $i \geq 0$; this ensures that the tensor product is a homologically graded curved Lie algebra. This convention becomes relevant within Section 4. Moreover, recall that given a pseudo-compact curved Lie algebra, $(\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}}$, and a unital cdga, A , both homologically graded their completed tensor product possesses a well defined pseudo-compact curved Lie algebra structure: if $\mathfrak{g} = \varprojlim_i \mathfrak{g}_i$, the completed tensor product is given by

$$\mathfrak{g} \hat{\otimes} A = \varprojlim_i \mathfrak{g}_i \otimes A,$$

where \otimes denotes the usual tensor product of dg vector spaces. More precisely, the curved Lie algebra structure is given by curvature $\omega_{\mathfrak{g}} \hat{\otimes} 1$, differential defined on elementary tensors by $d(x \hat{\otimes} a) = d_{\mathfrak{g}}x \hat{\otimes} a + (-1)^{|x|}x \hat{\otimes} d_A a$, and bracket defined on elementary tensors by $[x \hat{\otimes} a, y \hat{\otimes} b] = [x, y] \hat{\otimes} (-1)^{|y||a|}ab$. It will be common for the adjective ‘completed’ to be dropped and $\hat{\otimes}$ to be referred to as the tensor product.

The Lie algebras studied in this paper are often required to be pseudo-compact—particularly so for the main results. First recall that a pseudo-compact vector space is a linear topological vector which is complete¹ and possesses a topology generated by finite dimensional subspaces. Accordingly, finite dimensional pseudo-compact vector spaces are equipped with the discrete topology. This description easily extends to (differential) graded vector spaces. A curved Lie algebra is said to be pseudo-compact if it is an inverse limit of finite dimensional nilpotent curved Lie algebras, and as such is endowed with the natural topology induced from these finite dimensional subspaces. Moreover, the bracket and differential are required to be continuous with respect to this topology. This definition of a pseudo-compact curved Lie algebra is similar to the

¹A topological vector space has a unique uniform structure that is suitably well behaved with respect to the topology and so one can define completeness in the sense that all Cauchy sequences (or more generally Cauchy nets or filters) converge to a limit in the space.

definition of a pronilpotent Lie algebra, but more restrictive since every Lie algebra in the inverse system is assumed to be finite dimensional. Greater details concerning pseudo-compact algebras can be found, for example, in [Gar62, Section 3], [KY11, Appendix], and [VdB15, Section 4].

Although some of the Lie algebras in this paper will not necessarily be complexes (namely the curved Lie algebras), they resemble them and possess an odd derivation often referred to as the differential. In the homological grading, the derivation possessed by a curved Lie algebra has degree -1 . Given any homogeneous element, x , of some given graded algebra its degree is denoted by $|x|$. Therefore, in the homological setting a Maurer-Cartan element is of degree -1 and the curvature element is of degree -2 . Given a homologically graded space, V , define the suspension, ΣV , to be the homologically graded space using the convention $(\Sigma V)_i = V_{i-1}$. In the cohomological setting the suspension is defined by $(\Sigma V)^i = V^{i+1}$. When dealing with objects that are endowed with a topology (such as those that are pseudo-compact) taking the dual will be understood to mean taking the topological dual. In more detail, this is the functor that takes an object to the set of continuous morphisms from it to the ground field. Therefore, within this paper it will always be the case that $V^{**} \cong V$. Applying the functor of linear discrete (or topological) duality takes homologically graded spaces to cohomologically graded ones, and vice versa—more precisely, denoting the dual by an asterisk, it can be seen that $(V_i)^* = (V^*)^i$. Note, a homologically graded space can be made into a cohomologically graded one (and vice versa), by setting $V_i = V^{-i}$. Moreover, ΣV^* will be used to denote $\Sigma(V^*)$, and with this notation there exists an isomorphism $(\Sigma V)^* \cong \Sigma^{-1}V^*$.

Given a curved Lie algebra, $(\mathfrak{g}, d, \omega)$, a (descending) filtration of $(\mathfrak{g}, d, \omega)$ will be denoted by $F\mathfrak{g}$ and will correspond to the tower

$$\mathfrak{g} = F_1\mathfrak{g} \supseteq F_2\mathfrak{g} \supseteq F_3\mathfrak{g} \supseteq \dots$$

of subspaces $F_i\mathfrak{g}$ for all $i \in \mathbb{N}$. Notice here that only positively indexed filtrations are considered. The filtrations considered in this paper will respect the multiplication and differential, meaning

$$[F_i\mathfrak{g}, F_j\mathfrak{g}] \subseteq F_{i+j}\mathfrak{g} \qquad d(F_iA) \subseteq F_iA,$$

here the bracket and the differential are those of the curved Lie algebra $(\mathfrak{g}, d, \omega)$. A curved Lie algebra with a filtration is said to be a filtered curved Lie algebra.

Given a descending filtration $\{F_i\mathfrak{g}\}_{i \in \mathbb{N}}$ of a curved Lie algebra \mathfrak{g} , the associated graded algebra, denoted $\text{gr}_F \mathfrak{g}$, is the algebra given by the sum

$$\bigoplus_{i \in \mathbb{N}} \frac{F_i\mathfrak{g}}{F_{i+1}\mathfrak{g}}.$$

Notice that, since the filtrations respect the bracket and differential, the associated graded algebra inherits these operations.

Given a filtered pseudo-compact curved Lie algebra, \mathfrak{g} , its filtration is said to be complete if

$$\mathfrak{g} = \varprojlim_i \frac{\mathfrak{g}}{F_i\mathfrak{g}}.$$

Notice that the completeness condition is that of \mathfrak{g} being pronilpotent.

The filtration induced by the lower central series of a pseudo-compact curved Lie algebra \mathfrak{g} is the filtration

inductively given by

$$\begin{aligned}
F_1 \mathfrak{g} &= \mathfrak{g}, \\
F_2 \mathfrak{g} &= [F_1 \mathfrak{g}, \mathfrak{g}], \\
F_3 \mathfrak{g} &= [F_2 \mathfrak{g}, \mathfrak{g}], \\
&\vdots \\
F_{i+1} \mathfrak{g} &= [F_i \mathfrak{g}, \mathfrak{g}], \\
&\vdots
\end{aligned}$$

It will be shown in Proposition 1.9 that pseudo-compact curved Lie algebras are pronilpotent using the canonical filtration given by the lower central series. Therefore, the canonical filtration of a pseudo-compact curved Lie algebra, \mathfrak{g} , given by the lower central series is complete and thus Hausdorff. Recall that a filtration is said to be Hausdorff if

$$\bigcap_{i \geq 1} F_i A = 0.$$

Complete (and thus Hausdorff) filtrations that respect the bracket and differential with the additional condition that the associated graded objects have zero curvature (i.e. are complexes) are said to be admissible. For example, the filtration given by the lower central series of a curved Lie algebra is an admissible filtration, as shown by a combination of Proposition 1.9 and Proposition 1.10.

These constructions—where applicable—have a counterpart for cdgas whose definitions follow easily from those given above. It will only be necessary to consider filtrations of cdgas a couple of proofs, particular the proof of Lemma 3.16, whereas admissible filtrations play a crucial role in defining the weak equivalences of pseudo-compact curved Lie algebras. For more details regarding filtrations in a general context consult [Wei94].

1 The category of curved Lie algebras

In this section the category of pseudo-compact curved Lie algebras with curved morphisms will be introduced and some of its basic properties are discussed. Later, in Section 3.3, the category of pseudo-compact curved Lie algebras with curved morphisms will be shown to possess a model structure Quillen equivalent to the model structure for the category of unital cdgas given by Hinich [Hin97] via the adjoint functors given in Section 2.

Definition 1.1. *A curved Lie algebra is the triple $(\mathfrak{g}, d, \omega)$ where \mathfrak{g} is a graded Lie algebra, d is a derivation of \mathfrak{g} with $|d| = -1$ (known as the differential), and $\omega \in \mathfrak{g}$ with $|\omega| = -2$ (known as the curvature) such that:*

- $d \circ d(x) = [\omega, x]$, for all $x \in \mathfrak{g}$;
- $d\omega = 0$.

Remark 1.2. Notice that the differential is an abuse of notation as it need not square to zero. Some authors prefer to use the term predifferential. However, a curved Lie algebra with zero curvature is exactly a dgla.

Curved Lie algebras arise naturally when looking at twists of dglas; recall that given a dgla $(\mathfrak{g}, d, 0)$, considered here as a curved Lie algebra with zero curvature, define the twist of this dgla by some element $\xi \in \mathfrak{g}$ with $|\xi| = -1$, to be the curved Lie algebra $(\mathfrak{g}, d + ad_\xi, d\xi + \frac{1}{2}[\xi, \xi])$; here $ad_\xi(-) = [\xi, -]$ is the adjoint action. Now, this curved Lie algebra will have non-zero curvature if, and only if, the element ξ is not a solution to the Maurer-Cartan equation for the dgla $(\mathfrak{g}, d, 0)$. For more details of Lie algebra twists see [Bra12] or in the L_∞ -algebra case see [CL11].

It will be common practice within this paper to shorten the notation a curved Lie algebra, $(\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}})$, to only its underlying graded Lie algebra, \mathfrak{g} , where there is no ambiguity in doing so. In the shortened case, the differential and curvature will be denoted by the obvious subscript.

Recall that a curved Lie algebra is said to be pseudo-compact if it is an inverse limit of finite dimensional, nilpotent curved Lie algebras. Here it is important to note that there exists a natural topology on any given pseudo-compact curved Lie algebra (or more generally pseudo-compact vector space) which is induced by taking the inverse limit. For more details on pseudo-compact algebras see [Gar62, Section 3], [KY11, Appendix], and [VdB15, Section 4]. Therefore, when considering morphisms between pseudo-compact spaces they will always be assumed to be continuous with respect to this topology. Moreover, when taking the dual of such spaces it will be understood that it is the topological dual being taken as opposed to the linear discrete one.

The notion of a curved morphism will now be defined: these will make up the morphisms of the category of pseudo-compact curved Lie algebras.

Definition 1.3. *A curved morphism of curved Lie algebras is defined to be the pair*

$$(f, \alpha) : (\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}}) \rightarrow (\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}}),$$

where $f : \mathfrak{g} \rightarrow \mathfrak{h}$ is a morphism of graded Lie algebras and $\alpha \in \mathfrak{h}$ with $|\alpha| = -1$ such that:

- $d_{\mathfrak{h}}f(x) = f(d_{\mathfrak{g}}x) + [\alpha, f(x)]$, for all $x \in \mathfrak{g}$;
- $\omega_{\mathfrak{h}} = f(\omega_{\mathfrak{g}}) + d_{\mathfrak{h}}\alpha - \frac{1}{2}[\alpha, \alpha]$.

The image of an element $x \in \mathfrak{g}$ under the action of the curved morphism (f, α) is defined to be $f(x) - \alpha \in \mathfrak{h}$.

The composition of two curved morphisms, (f, α) and (g, β) , (when such a composition exists) is defined as follows:

$$(f, \alpha) \circ (g, \beta) = (f \circ g, \alpha + f(\beta)).$$

A morphism with $\alpha = 0$ is said to be strict.

Remark 1.4. Note that a curved morphism will map $0_{\mathfrak{g}} \mapsto -\alpha$.

Remark 1.5. In the case of a strict morphism it can be readily seen that the morphism is simply a dgla morphism where the image of the curvature of the domain is the curvature of the codomain: exactly like that of [CLM14]. Therefore, the α part of a curved morphism can be seen to act as an obstruction to the differentials commuting with the graded Lie algebra morphism and to the graded Lie algebra morphism preserving the curvature.

Proposition 1.6. *Given a morphism of curved Lie algebras, $(f, \alpha) : (\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}}) \rightarrow (\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}})$, there exists an inverse morphism $(f, \alpha)^{-1} : (\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}}) \rightarrow (\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}})$ such that:*

- $(f, \alpha) \circ (f, \alpha)^{-1} = (\text{id}_{\mathfrak{g}}, 0)$, and
- $(f, \alpha)^{-1} \circ (f, \alpha) = (\text{id}_{\mathfrak{h}}, 0)$,

if, and only if, f is an isomorphism of graded Lie algebras. Further, given an inverse graded Lie algebra morphism f^{-1} of f , the inverse of (f, α) is given by $(f^{-1}, -f^{-1}(\alpha))$.

Proof. If the graded Lie algebra morphism f is invertible, then clearly $(f^{-1}, -f^{-1}(\alpha))$ gives a two sided inverse of the morphism (f, α) . Conversely, if (f, α) is invertible with inverse $(f, \alpha)^{-1} = (g, \beta)$ then

- $(f, \alpha) \circ (f, \alpha)^{-1} = (f \circ g, f(\beta) + \alpha) = (\text{id}_{\mathfrak{g}}, 0)$, and
- $(f, \alpha)^{-1} \circ (f, \alpha) = (g \circ f, g(\alpha) + \beta) = (\text{id}_{\mathfrak{h}}, 0)$.

From these equations it is clear that g must be a two sided graded Lie algebra inverse for f . Additionally, it can easily be seen that $\beta = -g(\alpha)$. \square

Remark 1.7. It is important to note that a curved Lie algebra may be isomorphic to one with zero curvature (i.e. a dgla). To see this, let $ad_\xi(-) = [\xi, -]$ be the adjoint action and take the curved isomorphism

$$(\text{id}, \xi) : (\mathfrak{g}, d, \omega) \rightarrow \left(\mathfrak{g}, d + ad_\xi, \omega + d\xi + \frac{1}{2}[\xi, \xi] \right),$$

which has inverse $(\text{id}, -\xi)$. Now, noticing that the condition for ξ to be a Maurer-Cartan element of $(\mathfrak{g}, d, \omega)$ is precisely the curvature in the codomain. Whence, the resulting curved Lie algebra will have zero curvature if, and only if, the element ξ belongs to the set of Maurer-Cartan elements of $(\mathfrak{g}, d, \omega)$. In fact, these morphisms actually correspond to twisting by the element ξ ; such a twisting is denoted \mathfrak{g}^ξ ; see [Bra12] for more details, but note the notion of a curved morphism is not used therein.

Definition 1.8. *The category whose objects are pseudo-compact curved Lie algebras and morphisms are given by the continuous (with respect to the topology induced in taking the inverse limit) curved morphisms between them will be referred to as the category of pseudo-compact curved Lie algebras and will be denoted by $\hat{\mathcal{L}}$.*

In [BL05] it is shown that the functor of linear duality establishes an anti-equivalence between pseudo-compact Lie algebras and conilpotent Lie coalgebras, where pseudo-compact Lie algebras were referred to as pronilpotent Lie algebras. This term, however, may not be ideal as remarked in [LM15]. Despite this, pseudo-compact curved Lie algebras are pronilpotent in the classical sense.

Proposition 1.9. *For $\mathfrak{g} \in \hat{\mathcal{L}}$ let $\mathfrak{g} = \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \dots$ denote the lower central series. Then, $\mathfrak{g} = \varprojlim_i \frac{\mathfrak{g}}{\mathfrak{g}_i}$.*

Proof. Since $\mathfrak{g} \in \hat{\mathcal{L}}$, by definition $\mathfrak{g} = \varprojlim_n \mathfrak{g}^n$ where each \mathfrak{g}^n is a finite dimensional curved Lie algebra. Now, the filtered limit of finite dimensional vector spaces is exact leading to, for $i \geq 1$, $(\mathfrak{g}^n)_i \cong \varprojlim_n (\mathfrak{g}^n)_i$ and $\mathfrak{g}/\mathfrak{g}_i \cong \varprojlim_n \mathfrak{g}^n/(\mathfrak{g}^n)_i$. Since \mathfrak{g}^n is nilpotent, $\varprojlim_i \mathfrak{g}^n/(\mathfrak{g}^n)_i \cong \mathfrak{g}^n$. Whence,

$$\varprojlim_i \mathfrak{g}/\mathfrak{g}_i \cong \varprojlim_i \varprojlim_n \mathfrak{g}^n/(\mathfrak{g}^n)_i \cong \varprojlim_n \varprojlim_i \mathfrak{g}^n/(\mathfrak{g}^n)_i \cong \varprojlim_n \mathfrak{g}^n = \mathfrak{g},$$

as required. \square

The statement of Proposition 1.9 means that the filtration given by the lower central series is complete and Hausdorff. Moreover, it will be shown to be admissible. First, notice that the objects of $\hat{\mathcal{L}}$ do not form complexes (as it is not necessary that $d^2 = 0$) and as such there is no natural definition of a quasi-isomorphism. However, it is possible to define the notion of a filtered quasi-isomorphism. Filtered quasi-isomorphisms are of particular importance when defining the weak equivalences of the model structure in Section 3.3. First, an important observation must be made regarding the filtration given by the lower central series.

Proposition 1.10. *The canonical filtration given by the lower central series of a curved Lie algebra respects the differential (i.e. $d(F_n) \subseteq F_n$) and the bracket. Moreover, the associated graded curved Lie algebra will be a true complex in the fact that $d^2 = 0$.*

Proof. Respecting the differential and bracket are quick checks. To see that $d^2 = 0$ on the associated graded notice that the curvature is in F_1 trivially and thus taking the bracket of the curvature with anything will increase the filtration degree. \square

Since it has already been shown that the canonical filtration given by the lower central series of a curved Lie algebra is a Hausdorff filtration, it is now possible to conclude that it is an admissible filtration. Such filtrations are of great importance for the following reason.

Definition 1.11. *Given a morphism $(f, \alpha) : \mathfrak{g} \rightarrow \mathfrak{h}$ in $\hat{\mathcal{L}}$ and admissible filtrations, F , on \mathfrak{g} and \mathfrak{h} —i.e. that respect the brackets, respect the differentials, are complete, and are such that the associated graded objects $(\text{gr}_F \mathfrak{g}$ and $\text{gr}_F \mathfrak{h})$ are dgla—then $(f, \alpha) : \mathfrak{g} \rightarrow \mathfrak{h}$ is said to be a filtered quasi-isomorphism if the induced morphism $\text{gr}_F(f, \alpha) : \text{gr}_F \mathfrak{g} \rightarrow \text{gr}_F \mathfrak{h}$ is a quasi-isomorphism of dgla, i.e. induces an isomorphism on the level of homology.*

Remark 1.12. Notice that it is important that the filtrations be positively indexed, because even the seemingly innocent adaptation to allow non-negatively indexed filtrations could allow for the case when the filtrations are concentrated solely in degree 0 and in this case a filtered quasi-isomorphism would collapse to just a quasi-isomorphism. It is necessary that the notion of a weak equivalence in the model category of pseudo-compact curved Lie algebras is finer than a quasi-isomorphism, and so only positively indexed filtrations are considered.

It should be remarked here that no claim is made about the closure of filtered quasi-isomorphisms under composition.

2 Analogues of the Chevalley-Eilenberg and Harrison complexes

The category of unital cdgas with the standard dg algebra morphisms will be denoted by \mathcal{A} . Within this section a pair of functors will be defined (extending those of [LM15] and influenced by work of [Pos11]), before they are shown to be adjoint; providing the base for the Quillen equivalence proven in Theorem 3.19 of this paper.

In the next few paragraphs, a functor $\mathcal{L} : \mathcal{A} \rightarrow \hat{\mathcal{L}}$ will be constructed; several ideas of Positselski [Pos11] feature heavily in this construction. First, note that given a unital cdga, $A \in \mathcal{A}$, the underlying field is a subspace as $k = k \cdot 1 \subseteq A$, and therefore it is possible to take a linear retraction $\epsilon : A \rightarrow k$ (which may not commute with the differential or multiplication). Setting A_+ to be the kernel of this map, as vector spaces it is evident that $A = A_+ \oplus k$. Note, if A is augmented then ϵ can be chosen to be an augmentation (i.e. ϵ is a morphism of dg algebras) and A_+ is the augmentation ideal, i.e. the decomposition holds on the level of cdga. Note that because ϵ is not necessarily a morphism of dg algebras, the differential $d : A_+ \rightarrow A$ and the multiplication of $m : A_+ \otimes A_+ \rightarrow A$ can be split as $d = (d_+, d_k)$ and $m = (m_+, m_k)$. Where

$$\begin{aligned} d_+ : A_+ &\rightarrow A_+; & m_+ : A_+ \otimes A_+ &\rightarrow A_+; \\ d_k : A_+ &\rightarrow k; \text{ and} & m_k : A_+ \otimes A_+ &\rightarrow k. \end{aligned}$$

Definition 2.1. Given a unital cdga $A \in \mathcal{A}$ and a linear retraction $\epsilon : A \rightarrow k$ with kernel A_+ , let $\mathcal{L}(A)$ denote the pseudo-compact curved Lie algebra given by $(\hat{L}\Sigma A_+^*, d_+^* + m_+^*, d_k^* + m_k^*)$. Here the derivations d_+^* and m_+^* are the extensions of the duals to the whole curved Lie algebra (given by the Leibniz rule); the notation used here (somewhat abusively) is the same for both.

Remark 2.2. The pseudo-compact curved Lie algebra $\mathcal{L}(A)$ has zero curvature if the linear retraction ϵ is a morphism of unital cdga, i.e. if A is augmented with augmentation ϵ . This is because the k parts of the morphisms vanish.

A different choice of ϵ leads to isomorphic constructions: this is because a different choice is given by $\epsilon'(b) = \epsilon(b) + x$, where $x \in A_+$ has degree 0, and this leads to the isomorphism of constructions given by (id, x) (or a twisting), since x will have degree minus one in $\hat{L}(\Sigma A_+^*)$.

Let $A, B \in \mathcal{A}$, and A_+ and B_+ be the kernels of a pair of linear retractions on A and B respectively, then given a morphism $f : A \rightarrow B$ of \mathcal{A} the linear morphism $f : A_+ \rightarrow B$ can be split as $f = (f_+, f_k)$ where

$$\begin{aligned} f_+ : A_+ &\rightarrow B_+, \text{ and} \\ f_k : A_+ &\rightarrow k. \end{aligned}$$

Taking the duals of these morphisms, one obtains the linear morphisms:

$$\begin{aligned} f_+^* : B_+^* &\rightarrow A_+^*, \text{ and} \\ f_k^* : k &\rightarrow A_+^*. \end{aligned}$$

Clearly f_+^* can be extended to a graded Lie algebra morphism $f_+^* : \mathcal{L}(B) \rightarrow \mathcal{L}(A)$, denoted the same by an abuse of notation. Additionally, notice that it is possible to consider f_k^* as a degree -1 element of $\mathcal{L}(A)$. With these observations in mind the following proposition is made.

Proposition 2.3. *Given a morphism, $f : A \rightarrow B$ of \mathcal{A} , the morphism $(f_+^*, f_k^*) : \mathcal{L}(B) \rightarrow \mathcal{L}(A)$ constructed above is a well defined curved Lie algebra morphism.*

Proof. The proof amounts to chasing the definitions. □

Definition 2.4. *Let $\mathcal{L} : \mathcal{A} \rightarrow \hat{\mathcal{L}}$ be the functor that sends a cdga A to $\mathcal{L}(A)$ as in Definition 2.1 and sends a morphism f to (f_+^*, f_k^*) as in Proposition 2.3.*

Now to complete the pair it is necessary to describe a functor going in the reverse direction.

Definition 2.5. *The functor $\mathcal{C} : \hat{\mathcal{L}} \rightarrow \mathcal{A}$ is an analogue of the Chevalley-Eilenberg construction in homological algebra. That is, the underlying graded space is given by the symmetric algebra of the suspension of the continuous dual of \mathfrak{g} , i.e. $S\Sigma\mathfrak{g}^*$. This becomes a unital cdga with the concatenation product and the differential made of three parts coming from the duals of the curvature, the differential and the bracket of \mathfrak{g} made into derivations and extended via the Leibniz rule; c.f. [CLM14].*

Given a morphism $(f, \alpha) : \mathfrak{g} \rightarrow \mathfrak{h}$ in $\hat{\mathcal{L}}$ associate to it the morphism $\mathcal{C}(f, \alpha) : \mathcal{C}(\mathfrak{h}) \rightarrow \mathcal{C}(\mathfrak{g})$ given by $\Sigma(f^ \oplus \alpha^*) : \Sigma\mathfrak{h}^* \rightarrow \Sigma\mathfrak{g}^*$ extended as a morphism of cdgas.*

Notice that in Definition 2.5, the uncompleted symmetric algebra is taken and not the completed one. This is because \mathfrak{g} is already complete in some sense.

If $(\mathfrak{g}, d, 0)$ has zero curvature then the resulting cdga after applying the functor \mathcal{C} is augmented, c.f. [LM15] since it is then precisely the same construction. Therefore it can be understood that the curvature acts as an obstruction for the cdga $\mathcal{C}(\mathfrak{g})$ to be augmented, since the natural choice for augmentation fails to be a dg algebra morphism. More precisely, the part of the differential coming from the curvature maps into k and not \mathfrak{g} . Another reason to see why an augmentation fails to arise in the curved setting is that a MC element for a dgla corresponds to an augmentation of $S\Sigma\mathfrak{g}^*$; thus, in the uncurved case the zero element is always a MC element and there is always an augmentation. In the case of a curved Lie algebra, however, there need not be any solutions to the MC equation.

Proposition 2.6. *The functor $\mathcal{C} : \hat{\mathcal{L}} \rightarrow \mathcal{A}$ is right adjoint to the functor $\mathcal{L} : \mathcal{A} \rightarrow \hat{\mathcal{L}}$.*

Proof. In order to prove the proposition it is sufficient to exhibit the following isomorphism:

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{C}(\mathfrak{g}), A) \cong \mathrm{Hom}_{\hat{\mathcal{L}}}(\mathcal{L}(A), \mathfrak{g}),$$

for any curved Lie algebra \mathfrak{g} and any unital cdga A . To this end, assume

$$f : S\Sigma\mathfrak{g}^* \rightarrow A$$

is a morphism of unital cdgas. This morphism is uniquely determined by the linear morphism

$$f : \Sigma\mathfrak{g}^* \rightarrow A,$$

which in turn defines f_+ and f_k since $A = A_+ \oplus k$. Dualising, the linear morphisms

$$\begin{aligned} f_+^* : \Sigma A_+ &\rightarrow \mathfrak{g}, \text{ and} \\ f_k^* : \Sigma k &\rightarrow \mathfrak{g} \end{aligned}$$

are obtained.

By extending $f_+^* : \Sigma A_+ \rightarrow \mathfrak{g}$ as a graded Lie algebra morphism to $\hat{L}\Sigma A_+$ and combining it with f_k^* the curved Lie algebra morphism $(f_+^*, f_k^*) : \mathcal{L}(A) \rightarrow \mathfrak{g}$ is obtained. It is straightforward, although slightly tiresome, to do the calculations. Hence, one side of the adjunction is proven. Now, assume that

$$(f, \alpha) : \mathcal{L}(A) \rightarrow \mathfrak{g}$$

is a curved Lie algebra morphism. The graded Lie algebra morphism, f , is uniquely determined by the underlying linear morphism

$$f : \Sigma A_+^* \rightarrow \mathfrak{g}$$

which induces $f_+ : \Sigma \mathfrak{g}^* \rightarrow A_+$: this gives one component. The second comes from considering α as the morphism $\alpha : k \rightarrow \mathfrak{g}$ of degree minus two and dualising to obtain $f_k = \Sigma \alpha^* : \Sigma \mathfrak{g} \rightarrow k$. This construction yields a linear morphism $(f_+, f_k^*) : \Sigma \mathfrak{g} \rightarrow A$ which can be extended to morphism of commutative algebras

$$\mathcal{C}(\mathfrak{g}) \rightarrow A.$$

The final morphism also commutes with the differentials, which is a quick check. Hence the other side of the adjunction has been proven. \square

3 Model category of curved Lie algebras

Here it will be demonstrated that the category $\hat{\mathcal{L}}$ can be endowed with the structure of a model category with weak equivalences given by filtered quasi-isomorphisms. In addition, this model structure is Quillen equivalent to the model structure for unital cdga given in [Hin97]. This equivalence will be shown using similar methods to [LM15]. More precisely, by employing the Quillen equivalence that exists upon associative dg local algebras and pseudo-compact curved associative algebras (see [Pos11]), as well as the primitive elements and universal enveloping algebra functors. First, though, it must be proven that $\hat{\mathcal{L}}$ possesses all small limits and colimits.

3.1 Limits and colimits

Notice that there does not exist an initial object in the category of pseudo-compact curved Lie algebras with curved morphisms. The closest object to an initial object is the curved Lie algebra freely generated by a single element (the curvature) of degree -2 with zero differential, i.e. $(\hat{L}\langle\omega\rangle, 0, \omega)$. There clearly will be a morphism from this object to every other object. However, such a morphism is not necessarily unique. Nevertheless, in the category of curved Lie algebras with strict morphisms this object is the initial object. Thus, it is necessary to formally add an initial object to the category $\hat{\mathcal{L}}$. From here on let $\hat{\mathcal{L}}_*$ denote the category of pseudo-compact curved Lie algebras and curved morphisms with a formal initial object added. The category $\hat{\mathcal{L}}_*$ does possess a final object, namely the zero curved Lie algebra $(0, 0, 0)$.

In order to prove the category $\hat{\mathcal{L}}_*$ has all small limits, it suffices to prove that it has all products and equalisers, c.f. [ML98]. Likewise, to prove it has all small colimits it suffices to prove it has all coproducts and coequalisers.

Proposition 3.1. *Here the product over a finite set will be described; the general case follows in a straightforward fashion from this description. The product over the set $\{i_1, i_2, \dots, i_n\} = I$ indexing pseudo-compact curved Lie algebras, $(\mathfrak{g}_{i_j}, d_{i_j}, \omega_{i_j})$ for $1 \leq j \leq n$, is denoted by $\prod_{i \in I} \mathfrak{g}_i$ and given by the Cartesian product of underlying sets with bracket given by*

$$[(x_{i_1}, \dots, x_{i_n}), (x'_{i_1}, \dots, x'_{i_n})] = ([x_{i_1}, x'_{i_1}]_{i_1}, \dots, [x_{i_n}, x'_{i_n}]_{i_n}),$$

where $[\cdot]_{i_j}$ is the bracket of \mathfrak{g}_{i_j} , differential given by

$$d(x_{i_1}, \dots, x_{i_n}) = (d_{i_1} x_{i_1}, \dots, d_{i_n} x_{i_n}),$$

and curvature given by

$$(\omega_{i_1}, \dots, \omega_{i_n}).$$

The projection morphisms are the obvious ones onto each factor, i.e. $\pi_{i_j} : \prod_{i \in I} \mathfrak{g}_i \rightarrow \mathfrak{g}_{i_j}$.

\square

Proposition 3.2. *The equaliser of two curved morphisms $(f, \alpha), (g, \beta) : \mathfrak{g} \rightarrow \mathfrak{h}$ is the largest curved Lie subalgebra of \mathfrak{g} upon which the images of the curved morphisms agree, \mathfrak{e} , with the inclusion morphism $(\text{id}, 0) : \mathfrak{e} \rightarrow \mathfrak{g}$.*

□

Proposition 3.3. *The coproduct in the category of pseudo-compact curved Lie algebras is easiest to describe in the binary case: given two pseudo-compact curved Lie algebras, $(\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}})$ and $(\mathfrak{h}, d_{\mathfrak{h}}, \omega_{\mathfrak{h}})$, the coproduct $\mathfrak{g} \amalg \mathfrak{h}$ has the free Lie algebra on $\mathfrak{g}, \mathfrak{h}$ and a formal element x of degree minus one as the underlying graded Lie algebra. The differential is given by the rules: $d|_{\mathfrak{g}} = d_{\mathfrak{g}}, d|_{\mathfrak{h}} = d_{\mathfrak{h}} - \text{ad}_x$ and $dx = \omega_{\mathfrak{h}} - \omega_{\mathfrak{g}} - \frac{1}{2}[x, x]$. The resulting space has curvature equal to that of \mathfrak{g} .*

Proof. First it is necessary to check that the resulting space is in fact a curved Lie algebra by checking the equations in Definition 1.1. Clearly it is sufficient to show they hold for the generators. Now, for $a \in \mathfrak{g}$

$$\begin{aligned} d_{\mathfrak{g} \amalg \mathfrak{h}} \circ d_{\mathfrak{g} \amalg \mathfrak{h}} a &= d_{\mathfrak{g}} \circ d_{\mathfrak{g}} a \\ &= [\omega_{\mathfrak{g}}, a], \end{aligned}$$

and for $b \in \mathfrak{h}$

$$\begin{aligned} d_{\mathfrak{g} \amalg \mathfrak{h}} \circ d_{\mathfrak{g} \amalg \mathfrak{h}} b &= d_{\mathfrak{g} \amalg \mathfrak{h}} \circ d_{\mathfrak{h}} b - d_{\mathfrak{g} \amalg \mathfrak{h}} [x, b] \\ &= [\omega_{\mathfrak{h}}, b] - [x, d_{\mathfrak{h}} b] - [d_{\mathfrak{g} \amalg \mathfrak{h}} x, b] - (-1)^{|x|} [x, d_{\mathfrak{g} \amalg \mathfrak{h}} b] \\ &= [\omega_{\mathfrak{h}}, b] - [x, d_{\mathfrak{h}} b] - \left[\omega_{\mathfrak{h}} - \omega_{\mathfrak{g}} - \frac{1}{2}[x, x], b \right] + [x, d_{\mathfrak{h}} b] - [x, [x, b]] \\ &= [\omega_{\mathfrak{g}}, b] + \frac{1}{2}[[x, x], b] - [x, [x, b]] \\ &= [\omega_{\mathfrak{g}}, b], \end{aligned}$$

where the final step is due to the graded Jacobi identity. Finally,

$$\begin{aligned} d_{\mathfrak{g} \amalg \mathfrak{h}} \circ d_{\mathfrak{g} \amalg \mathfrak{h}} x &= d_{\mathfrak{g} \amalg \mathfrak{h}} \left(\omega_{\mathfrak{h}} - \omega_{\mathfrak{g}} - \frac{1}{2}[x, x] \right) \\ &= -[x, \omega_{\mathfrak{h}}] - \frac{1}{2}([dx, x] - [x, dx]) \\ &= [\omega_{\mathfrak{h}}, x] - [dx, x] \\ &= [\omega_{\mathfrak{g}}, x] + \frac{1}{2}[[x, x], x] \\ &= [\omega_{\mathfrak{g}}, x]. \end{aligned}$$

Clearly, $d_{\mathfrak{g} \amalg \mathfrak{h}} \omega_{\mathfrak{g}} = 0$ and thus $\mathfrak{g} \amalg \mathfrak{h}$ is in fact a curved Lie algebra. What remains to be shown is that it possesses the required universal property, i.e. given a curved Lie algebra X and curved morphisms $(f_{\mathfrak{g}}, \alpha) : \mathfrak{g} \rightarrow X$ and $(f_{\mathfrak{h}}, \beta) : \mathfrak{h} \rightarrow X$ there exists a unique morphism $(f, \gamma) : \mathfrak{g} \amalg \mathfrak{h} \rightarrow X$ making the diagram

$$\begin{array}{ccccc} & & X & & \\ & \nearrow (f_{\mathfrak{g}}, \alpha) & \uparrow & \nwarrow (f_{\mathfrak{h}}, \beta) & \\ \mathfrak{g} & \longrightarrow & \mathfrak{g} \amalg \mathfrak{h} & \longleftarrow & \mathfrak{h} \end{array}$$

commute. Clearly, when restricting to the underlying graded algebras it is possible to take $f|_{\mathfrak{g}} = f_{\mathfrak{g}}$ and $f|_{\mathfrak{h}} = f_{\mathfrak{h}}$. Further, one can set $f(x) = \beta - \alpha$. Hence, taking the curved morphism as (f, α) for $a \in \mathfrak{g}$ the equation

$$\begin{aligned} d_X f(a) &= f d_{\mathfrak{g}} a + [\alpha, f(a)] \\ &= f_{\mathfrak{g}} d_{\mathfrak{g}} a + [\alpha, f_{\mathfrak{g}}(a)], \end{aligned}$$

and for $b \in \mathfrak{h}$ the equation

$$\begin{aligned} d_X f(b) &= f d_{\mathfrak{h}} b - [f(x), f(b)] + [\alpha, f(b)] \\ &= f_{\mathfrak{h}} d_{\mathfrak{h}} b - [\alpha - \beta, f_{\mathfrak{h}}(b)] + [\alpha, f_{\mathfrak{h}}(b)] \\ &= f_{\mathfrak{h}} d_{\mathfrak{h}} b + [\beta, f_{\mathfrak{h}}(b)] \end{aligned}$$

both hold since $(f_{\mathfrak{g}}, \alpha) : \mathfrak{g} \rightarrow X$ and $(f_{\mathfrak{h}}, \beta) : \mathfrak{h} \rightarrow X$ are curved. Additionally, the equation regarding the curvature holds trivially since it is precisely the one for $f_{\mathfrak{g}}$. Uniqueness is a quick check. \square

Remark 3.4. Fix $\mathfrak{h} \in \hat{\mathcal{L}}_*$. Given $\mathfrak{g} \in \hat{\mathcal{L}}_*$, the functor assigning $\mathfrak{g} \mapsto \mathfrak{g} \amalg \mathfrak{h}$ is not exact. This is easily seen as the terminal object is not preserved: $0 \mapsto (\mathfrak{h}\langle x \rangle)^x$ which is clearly non-zero.

Remark 3.5. Since \amalg is a coproduct in the category $\hat{\mathcal{L}}_*$ there exists a curved isomorphism

$$\mathfrak{g} \amalg \mathfrak{h} \cong \mathfrak{h} \amalg \mathfrak{g}.$$

Here this isomorphism will be explicitly constructed. The isomorphism is strictly curved as the two resulting curved Lie algebras are related by a twist.

Beginning with $\mathfrak{g} \amalg \mathfrak{h}$ which has underlying space $\mathfrak{g} * \mathfrak{h} * \langle x \rangle$ (the same as the target space), on the level of graded spaces it is mapped to the resulting space by the graded Lie algebra morphism that restricts to the identity on \mathfrak{g} and \mathfrak{h} whereas sends x to $-x$. Denote this graded Lie algebra morphism by f ; it is clearly an isomorphism on the underlying spaces, being its own inverse. Having explained how the underlying graded Lie algebras are related, it remains to show how the differential structures are related: they are related by a twisting by the element $-x$. This acts on the differentials as follows

$$\begin{aligned} d|_{\mathfrak{g}} &= d_{\mathfrak{g}} \mapsto d_{\mathfrak{g}} - ad_x \\ d|_{\mathfrak{h}} &= d_{\mathfrak{h}} - ad_x \mapsto d_{\mathfrak{h}} + ad_x - ad_x = d_{\mathfrak{h}} \\ dx &= \omega_{\mathfrak{h}} - \omega_{\mathfrak{g}} - \frac{1}{2}[x, x] \mapsto dx = \omega_{\mathfrak{g}} - \omega_{\mathfrak{h}} + \frac{1}{2}[x, x] - ad_x = \omega_{\mathfrak{g}} - \omega_{\mathfrak{h}} - \frac{1}{2}[x, x]. \end{aligned}$$

Therefore, the isomorphism is $(f, -x) : \mathfrak{g} \amalg \mathfrak{h} \rightarrow \mathfrak{h} \amalg \mathfrak{g}$ with inverse $(f, -x) : \mathfrak{h} \amalg \mathfrak{g} \rightarrow \mathfrak{g} \amalg \mathfrak{h}$.

The (categorical) coproduct of Propositions 3.3 is similar to the (non-categorical) disjoint product of [LM15]; it can be informally thought of as taking the disjoint union of the two spaces, formally adding a MC basepoint (i.e. a solution the Maurer-Cartan equation) and then twisting the copy of \mathfrak{h} with this basepoint to flatten its curvature.

Proposition 3.6. *The coequaliser of two curved morphisms $(f, \alpha), (g, \beta) : \mathfrak{g} \rightarrow \mathfrak{h}$ is the quotient of \mathfrak{h} by the ideal generated by $f(x) - g(x)$ and $\alpha - \beta$, for all $x \in \mathfrak{g}$.*

\square

Therefore, it can be seen that $\hat{\mathcal{L}}_*$ has all small limits and colimits.

3.2 DG duality for associative algebras

It is now required that the definitions of the cobar constructions in the associative case be recalled, found for example in [Pos11] where pseudo-compact local associative algebras were studied in the dual setting as conilpotent coassociative coalgebras. Let $\mathcal{A}ss$ denote the category of associative dg algebras with dg algebra morphisms and $\widehat{\mathcal{A}ss}$ denote the category of pseudo-compact local associative curved algebras with continuous curved associative algebra morphisms. Recall that a curved associative algebra is a graded algebra with an odd derivation and a distinguished curvature element, satisfying similar axioms to a curved Lie algebra. The two categories $\mathcal{A}ss$ and $\widehat{\mathcal{A}ss}$ possess model structures, c.f. [Pos11]: the weak equivalences of $\mathcal{A}ss$ are the quasi-isomorphisms and the weak equivalences of $\widehat{\mathcal{A}ss}$ are those belonging to the minimal class of morphisms that are filtered quasi-isomorphisms and satisfy the two out of three property.

Definition 3.7. Let $\hat{\mathcal{B}} : \mathcal{A}ss \rightarrow \widehat{\mathcal{A}ss}$ be the functor assigning to an associative dg algebra the pseudo-compact associative curved algebra $\hat{\mathcal{B}}(A)$ whose underlying graded algebra is $\hat{T}\Sigma A_+^*$, where A_+ is the kernel of a linear retraction $A \rightarrow k$. The differential is induced from the multiplication and differential in the same way as Definition 2.5. The resulting pseudo-compact associative curved algebra is uncurved (i.e. has zero curvature) if, and only if, the linear retraction is a true augmentation, i.e. a dg algebra morphism.

Let $\mathcal{B} : \widehat{\mathcal{A}ss} \rightarrow \mathcal{A}ss$ be the functor assigning to a pseudo-compact associative curved algebra the (discrete) associative dg algebra $\mathcal{B}(A)$ whose underlying graded algebra is $T\Sigma A_+^*$, where A_+ is again the kernel of a linear retraction $A \rightarrow k$. The differential is induced in the same way as in Definition 2.4.

Much like in [LM15], the reason for recalling the definitions of the associative case is that the functors \mathcal{C} and \mathcal{L} can be ‘embedded’ into the following adjunction proven by Positselski [Pos11].

Theorem 3.8. The functors \mathcal{B} and $\hat{\mathcal{B}}$ are adjoint. Moreover, they induce a Quillen equivalence.

Proof. One must simply notice that pseudo-compact local associative curved algebras are dual to conilpotent coassociative curved coalgebras and thus it follows from [Pos11]. \square

The functors \mathcal{L} and \mathcal{C} can be seen to fit into the following commutative diagram of functors:

$$\begin{array}{ccc}
 \mathcal{A}ss & \xrightleftharpoons[\mathcal{B}]{\hat{\mathcal{B}}} & \widehat{\mathcal{A}ss} \\
 \text{forgetful} \downarrow & & \downarrow U \\
 \hat{\mathcal{L}}_* & \xrightleftharpoons[\mathcal{C}]{\mathcal{L}} & \mathcal{A}
 \end{array}
 ,$$

where $U : \hat{\mathcal{L}}_* \rightarrow \widehat{\mathcal{A}ss}$ is the universal enveloping algebra construction and $\text{Prim} : \widehat{\mathcal{A}ss} \rightarrow \hat{\mathcal{L}}_*$ is the primitive elements functor. The following two propositions show commutativity (up to weak equivalence) of this diagram and the proofs are the same as [LM15].

Proposition 3.9. Given a unital cdga, A , there is a natural isomorphism of pseudo-compact curved Lie algebras:

$$\text{Prim}(\hat{\mathcal{B}}(A)) \cong \mathcal{L}(A).$$

\square

Proposition 3.10. Given a pseudo-compact curved Lie algebra, \mathfrak{g} , there is a quasi-isomorphism of differential graded algebras

$$\mathcal{B}(U(\mathfrak{g})) \simeq \mathcal{C}(\mathfrak{g}).$$

\square

3.3 Model structure

The category $\hat{\mathcal{L}}_*$ will now be endowed with a model structure that will be shown to be Quillen equivalent (via the functors defined in Section 2) to the model category of unital cdgas given by Hinich [Hin97].

Definition 3.11. A morphism $(f, \alpha) : \mathfrak{g} \rightarrow \mathfrak{h}$ in $\hat{\mathcal{L}}_*$ is called

- a weak equivalence if, and only if, it belongs to the minimal class of filtered quasi-isomorphisms that satisfy the two out of three property;
- a fibration if, and only if, the underlying graded Lie algebra morphism, f , is a surjective morphism;
- a cofibration if, and only if, it has the LLP with respect to all acyclic fibrations.

Provided with this definition, some preliminary results will be discussed before showing that $\hat{\mathcal{L}}_*$ is in fact a model category with this model structure. First it is helpful to look at some useful facts regarding filtered quasi-isomorphisms and the units of the adjunction given in Section 2.

Proposition 3.12. Given a filtered quasi-isomorphism $(f, \alpha) : \mathfrak{g} \rightarrow \mathfrak{h}$, the induced morphism

$$\mathcal{C}(f, \alpha) : \mathcal{C}(\mathfrak{h}) \rightarrow \mathcal{C}(\mathfrak{g})$$

is a quasi-isomorphism of \mathcal{A} . Conversely, given a quasi-isomorphism $g : A \rightarrow B$ of \mathcal{A} the induced morphism

$$\mathcal{L}(g) : \mathcal{L}(B) \rightarrow \mathcal{L}(A)$$

is a filtered quasi-isomorphism.

Proof. In the first instance, there exist filtrations, F , on \mathfrak{g} and \mathfrak{h} such that $\text{gr}_F(f, \alpha)$ is a quasi-isomorphism. These filtrations induce filtrations upon $\mathcal{C}(\mathfrak{g})$ and $\mathcal{C}(\mathfrak{h})$. Therefore, $\text{gr}_F \mathcal{C}(f, \alpha)$ is a quasi-isomorphism and hence $\mathcal{C}(f, \alpha)$ is a quasi-isomorphism. Now for the converse statement; after applying the functor \mathcal{L} , take the filtration induced by maximal bracket length and since these brackets are built freely they preserve quasi-isomorphisms whence $\mathcal{L}(g)$ is a filtered quasi-isomorphism. \square

Proposition 3.13.

- Given a unital cdga, A , the morphism $i_A : \mathcal{C}\mathcal{L}(A) \rightarrow A$ is a quasi-isomorphism.
- Given a pseudo-compact curved Lie algebra, \mathfrak{g} , the morphism $i_{\mathfrak{g}} : \mathcal{L}\mathcal{C}(\mathfrak{g}) \rightarrow \mathfrak{g}$ is a weak equivalence of curved Lie algebras, i.e. it is a filtered quasi-isomorphism.

Proof. By Proposition 3.10 $\mathcal{C}(\mathcal{L}(A)) \simeq \mathcal{B}\mathcal{U}(\mathcal{L}(A))$. Now, $\mathcal{B}\mathcal{U}\mathcal{L}(A) = \mathcal{B}\hat{\mathcal{B}}(A)$ which is quasi-isomorphic to A by [Pos11], considering A as an associative dg algebra. Thus the first part is proven.

To prove the second statement, consider the natural filtration by bracket length on \mathfrak{g} and the filtration it induces upon $\mathcal{L}\mathcal{C}(\mathfrak{g})$; denote these two filtrations by F . Therefore, it is sufficient to show that the morphism $\mathcal{L}\mathcal{C}(\text{gr}_F \mathfrak{g}) \rightarrow \text{gr}_F \mathfrak{g}$ is a quasi-isomorphism of dgla—this follows from [LM15]. \square

Lemma 3.14. The functor \mathcal{L} sends fibrations to cofibrations and cofibrations to fibrations.

Proof. Given a fibration $f : A \rightarrow B$ of unital cdgas, to show that $\mathcal{L}(f)$ is a cofibration it is necessary to find a lift in the each diagram of the form

$$\begin{array}{ccc} \mathcal{L}(B) & \longrightarrow & \mathfrak{g} \\ \mathcal{L}(f) \downarrow & \nearrow & \downarrow (\phi, \alpha) \\ \mathcal{L}(A) & \longrightarrow & \mathfrak{h} \end{array},$$

where the morphism $(\phi, \alpha) : \mathfrak{g} \rightarrow \mathfrak{h}$ is an acyclic fibration of pseudo-compact curved Lie algebras. This is equivalent to seeking a lift in each diagram of the following form

$$\begin{array}{ccc} \mathcal{C}(\mathfrak{h}) & \longrightarrow & A \\ \mathcal{C}(\phi, \alpha) \downarrow & \nearrow & \downarrow f \\ \mathcal{C}(\mathfrak{g}) & \longrightarrow & B \end{array},$$

having used the adjunction of the functors \mathcal{L} and \mathcal{C} . Now, by assumption f is a fibration and $\mathcal{C}(\phi, \alpha)$ is a weak equivalence by Proposition 3.12. Therefore, it suffices show that the morphism $\mathcal{C}(\phi, \alpha)$ is a cofibration. Looking at the kernel of (ϕ, α) a subspace $K \subseteq \mathfrak{g}$ and a tower

$$\mathfrak{h} \cong \mathfrak{g}/(\mathfrak{g}_1 \cap K) \xleftarrow{\pi_1} \mathfrak{g}/(\mathfrak{g}_2 \cap K) \xleftarrow{\pi_2} \mathfrak{g}/(\mathfrak{g}_3 \cap K) \xleftarrow{\pi_3} \dots$$

are obtained. In Proposition 1.9 it was shown for $\mathfrak{g} \in \hat{\mathcal{L}}_*$ that $\mathfrak{g} = \varprojlim_i \mathfrak{g}/\mathfrak{g}_i$ where $\{\mathfrak{g}_i\}_{i \in \mathbb{N}}$ denotes the lower central series. Therefore, the limit of the above tower is simply \mathfrak{g} and hence the colimit of

$$\mathcal{C}(\mathfrak{h}) \hookrightarrow \mathcal{C}(\mathfrak{g}/(\mathfrak{g}_1 \cap K)) \hookrightarrow \mathcal{C}(\mathfrak{g}/(\mathfrak{g}_2 \cap K)) \hookrightarrow \mathcal{C}(\mathfrak{g}/(\mathfrak{g}_3 \cap K)) \hookrightarrow \dots$$

is $\mathcal{C}(\mathfrak{g})$. It is thus sufficient to prove that each of the morphisms $\mathcal{C}(\pi_n)$ for $n \in \mathbb{N}$ are cofibrations in \mathcal{A} . First note that for each $n \geq 1$, $\ker(\pi_n) = (\mathfrak{g}_n \cap K)/(\mathfrak{g}_{n+1} \cap K)$. From here it is straightforward to see that, like in 5.2.2 of [Hin01], that $\mathcal{C}(\pi_n)$ is a standard cofibration obtained by adding free generators to $\mathcal{C}(\mathfrak{g}/(\mathfrak{g}_n \cap K))$.

To complete the proof of the statement, note the standard fact that cofibrations in unital cdga are monomorphisms and the functor \mathcal{L} sends monomorphisms to epimorphisms, i.e. fibrations. \square

By Theorem 9.7 of [DS95], once it is shown that $\hat{\mathcal{L}}_*$ is a CMC with the model structure of Definition 3.11, Theorem 3.19 will have been proven. To this end, first it is noted that despite the functor $(-) \amalg \mathfrak{h}$, for some fixed \mathfrak{h} , not being exact (see Remark 3.4) it does have the following redeeming property.

Lemma 3.15. *Given a weak equivalence, $(f, \alpha) : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ of $\hat{\mathcal{L}}_*$, for a fixed \mathfrak{h}*

$$(f, \alpha) \amalg (\text{id}_{\mathfrak{h}}, 0) : \mathfrak{g}_1 \amalg \mathfrak{h} \rightarrow \mathfrak{g}_2 \amalg \mathfrak{h}$$

is a weak equivalence too.

Proof. Since the morphism (f, α) is a weak equivalence, there exists filtrations on \mathfrak{g}_1 and \mathfrak{g}_2 such that the induced morphism on the associated graded algebras is a quasi-isomorphism. These filtrations induce filtrations on the coproducts with the fixed curved Lie algebra \mathfrak{h} , and the associated graded algebras are clearly quasi-isomorphic via the induced morphisms. \square

Now, enough auxiliary results have been developed to prove the remaining axioms of a model category, i.e. the lifting and the factorisation properties. The proofs of the lifting axioms rely on the next lemma which is proven with methods based upon that of [Pos11] which in turn are based upon constructions originally performed in [Hin01] in the proof of a similar lemma named op. cit. the ‘Key Lemma’.

Lemma 3.16. *Let A be a unital cdga, \mathfrak{g} be a pseudo-compact curved Lie algebra and $f : A \rightarrow \mathcal{C}(\mathfrak{g})$ be a surjective morphism. Consider the pushout*

$$\begin{array}{ccc}
\mathcal{LC}(\mathfrak{g}) & \xrightarrow{\mathcal{L}(f)} & \mathcal{L}(A) \\
i_{\mathfrak{g}} \downarrow & & \downarrow j \\
\mathfrak{g} & \longrightarrow & \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}
\end{array}$$

Then the morphism $j : \mathcal{L}(A) \rightarrow \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}$ is weak equivalence.

Proof. First, consider the two spaces in question as non-differential graded algebras. Since the morphism $f : A \rightarrow \mathcal{C}(\mathfrak{g})$ is surjective,

$$A \cong \mathcal{C}(\mathfrak{g}) \oplus \ker(A \rightarrow \mathcal{C}(\mathfrak{g}))$$

as graded algebras and hence as graded Lie algebras

$$\mathcal{L}(A) \cong \mathcal{LC}(\mathfrak{g}) \coprod \mathcal{L}(\ker(A \rightarrow \mathcal{C}(\mathfrak{g}))).$$

This leads to the isomorphism

$$\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g} \cong \mathcal{LC}(\mathfrak{g}) \coprod \mathcal{L}(\ker(A \rightarrow \mathcal{C}(\mathfrak{g}))).$$

Hence, since $\mathcal{LC}(\mathfrak{g})$ and \mathfrak{g} are weakly equivalent, as non-differential spaces $\mathcal{L}(A)$ and $\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}$ are weakly equivalent. However, it does not immediately follow that this remains true when adding the differentials to the spaces. In order to show the lemma is in fact true, it suffices to show that j is a filtered quasi-isomorphism by choosing some suitable filtrations.

Filter \mathfrak{g} by the natural filtration obtained by the lower central series. Denote this filtration, that induced upon $\mathcal{C}(\mathfrak{g})$ and that induced on A by the pre-images of the surjective morphism $A \rightarrow \mathcal{C}(\mathfrak{g})$ by F . Therefore, there exists a morphism $\mathcal{LC}(\mathrm{gr}_F \mathfrak{g}) \rightarrow \mathcal{L}(\mathrm{gr}_F A)$ which is a cofibration. So it is clear that $\mathrm{gr}_F(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}) = \mathcal{L}(\mathrm{gr}_F A) \coprod_{\mathcal{LC}(\mathrm{gr}_F \mathfrak{g})} \mathrm{gr}_F \mathfrak{g}$, i.e. take the pushout:

$$\begin{array}{ccc}
\mathcal{LC}(\mathrm{gr}_F \mathfrak{g}) & \longrightarrow & \mathcal{L}(\mathrm{gr}_F A) \\
\downarrow & & \downarrow \\
\mathrm{gr}_F \mathfrak{g} & \longrightarrow & \mathrm{gr}_F(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g})
\end{array}$$

Denote by n the positive grading induced by the indexing of the filtration F and introduce a filtration G on $\mathrm{gr}_F A$ by setting $G_0 \mathrm{gr}_F A = \mathrm{gr}_F A$ and G_j the sum of the components of the ideal $\ker(\mathrm{gr}_F A \rightarrow \mathcal{C}(\mathrm{gr}_F \mathfrak{g}))$ situated in the grading $n \geq j$. This would mean that the associated graded is equal to

$$\begin{aligned}
\mathrm{gr}_G \mathrm{gr}_F A &= \frac{\mathrm{gr}_F A}{\oplus_{n \geq 1} (\ker(\mathrm{gr}_F A \rightarrow \mathcal{C}(\mathrm{gr}_F \mathfrak{g})))_n} \oplus \frac{\oplus_{n \geq 1} (\ker(\mathrm{gr}_F A \rightarrow \mathcal{C}(\mathrm{gr}_F \mathfrak{g})))_n}{\oplus_{n \geq 2} (\ker(\mathrm{gr}_F A \rightarrow \mathcal{C}(\mathrm{gr}_F \mathfrak{g})))_n} \\
&\quad \oplus \frac{\oplus_{n \geq 2} (\ker(\mathrm{gr}_F A \rightarrow \mathcal{C}(\mathrm{gr}_F \mathfrak{g})))_n}{\oplus_{n \geq 3} (\ker(\mathrm{gr}_F A \rightarrow \mathcal{C}(\mathrm{gr}_F \mathfrak{g})))_n} \oplus \dots \\
&= \mathcal{C}(\mathrm{gr}_F \mathfrak{g}) \oplus (\ker(\mathrm{gr}_F A \rightarrow \mathcal{C}(\mathrm{gr}_F \mathfrak{g})))_1 \oplus (\ker(\mathrm{gr}_F A \rightarrow \mathcal{C}(\mathrm{gr}_F \mathfrak{g})))_2 \oplus \dots
\end{aligned}$$

This grading is locally finite with respect to the grading n , but the differential may map $\mathrm{gr}_F A$ into the kernel of $\mathrm{gr}_F A \rightarrow \mathcal{C}(\mathrm{gr}_F \mathfrak{g})$, and so the goal of a filtered quasi-isomorphism has yet been achieved. Let G also denote the induced filtration upon $\mathcal{L}(\mathrm{gr}_F A)$ and hence $\mathcal{L}(\mathrm{gr}_F A) \coprod_{\mathcal{LC}(\mathrm{gr}_F \mathfrak{g})} \mathrm{gr}_F \mathfrak{g}$. Whence

$$\mathrm{gr}_G \mathrm{gr}_F \left(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g} \right) = \mathcal{L}(\mathrm{gr}_G \mathrm{gr}_F A) \coprod_{\mathcal{LC}(\mathrm{gr}_F \mathfrak{g})} \mathrm{gr}_F \mathfrak{g}.$$

Therefore, there are two gradings, n and j , upon $\text{gr}_G \text{gr}_F A$ and $\text{gr}_G \text{gr}_F(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g})$. Thus, introduce a final filtration H by the rules $H_t \text{gr}_G \text{gr}_F A = \bigoplus_{n \geq 1, j \geq t} (\text{gr}_G \text{gr}_F A)_{n,j}$. This again induces a filtration upon $\mathcal{L}(\text{gr}_F A) \coprod_{\mathcal{LC}(\text{gr}_F \mathfrak{g})} \mathfrak{g}$. Whence

$$\text{gr}_H \text{gr}_G \text{gr}_F \left(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g} \right) = \mathcal{L}(\text{gr}_H \text{gr}_G \text{gr}_F A) \coprod_{\mathcal{LC}(\text{gr}_F \mathfrak{g})} \text{gr}_F \mathfrak{g}.$$

Now, $\mathcal{L}(\text{gr}_H \text{gr}_G \text{gr}_F A) = \mathcal{LC}(\text{gr}_F \mathfrak{g}) \coprod X$ where X is constructed from applying the functor \mathcal{L} to the kernel components of $\text{gr}_G \text{gr}_F(A)$ and $\text{gr}_H \text{gr}_G \text{gr}_F(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}) = \text{gr}_F \mathfrak{g} \coprod X$. Further, since $\mathcal{LC}(\text{gr}_F \mathfrak{g}) \rightarrow \text{gr}_F \mathfrak{g}$ is a quasi-isomorphism, so is $\mathcal{L}(\text{gr}_H \text{gr}_G \text{gr}_F A) \rightarrow \text{gr}_H \text{gr}_G \text{gr}_F(\mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g})$ and hence the statement has been proven. \square

Lemma 3.17. *Given a morphism in the category of curved Lie algebras, $(f, \alpha) : \mathfrak{g} \rightarrow \mathfrak{h}$, it can be factorised as the composition of*

- *a fibration followed by an acyclic fibration; and*
- *an acyclic cofibration followed by a fibration.*

Proof. To proceed, first consider the induced morphism $\mathcal{C}(f, \alpha) : \mathcal{C}(\mathfrak{h}) \rightarrow \mathcal{C}(\mathfrak{g})$. Since \mathcal{A} is a model category it is possible to factorise this morphism as

$$\mathcal{C}(\mathfrak{h}) \xrightarrow{i} A \xrightarrow{p} \mathcal{C}(\mathfrak{g}),$$

where i is a cofibration and p is a fibration in the category of unital cdga. Further, it is possible to choose one of the morphisms, i and p , to be a weak equivalence and doing so will specialise to one of the statements in the lemma. This factorisation in turn induces morphisms

$$\mathcal{LC}(\mathfrak{g}) \xrightarrow{\mathcal{L}(p)} \mathcal{L}(A) \xrightarrow{\mathcal{L}(i)} \mathcal{LC}(\mathfrak{h})$$

of curved Lie algebras. Therefore, taking the pushout

$$\begin{array}{ccc} \mathcal{LC}(\mathfrak{g}) & \xrightarrow{\mathcal{L}(p)} & \mathcal{L}(A) \\ i_{\mathfrak{g}} \downarrow & & \downarrow \tilde{i} \\ \mathfrak{g} & \xrightarrow{\iota} & \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g} \end{array},$$

it can be seen that the morphism $\iota : \mathfrak{g} \rightarrow \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g}$ is a cofibration, since it is obtained from a cobase change of the cofibration $\mathcal{L}(p)$. Further, there exists a morphism $\tilde{p} : \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g} \rightarrow \mathfrak{h}$ coming from the universal property of a pushout:

$$\begin{array}{ccc} \mathcal{LC}(\mathfrak{g}) & \xrightarrow{\mathcal{L}(p)} & \mathcal{L}(A) \\ i_{\mathfrak{g}} \downarrow & & \downarrow \tilde{i} \\ \mathfrak{g} & \xrightarrow{\iota} & \mathcal{L}(A) \coprod_{\mathcal{LC}(\mathfrak{g})} \mathfrak{g} \end{array} \begin{array}{c} \xrightarrow{i_{\mathfrak{h}} \circ \mathcal{L}(i)} \\ \xrightarrow{\tilde{p}} \\ \xrightarrow{f} \end{array} \mathfrak{h},$$

and hence \tilde{p} is surjective. Further, Lemma 3.16 shows that \tilde{i} is a weak equivalence. The composition $f = \tilde{p} \circ \iota$ provides the desired decompositions depending upon the choice of weak equivalence in the original decomposition. To see this for the first statement, it is necessary to show that \tilde{p} is a weak equivalence. It is, therefore, sufficient to show that $i_{\mathfrak{h}} \circ \mathcal{L}(i)$ is a weak equivalence since it then follows from the two of three property. To this end, choose i in the original factorisation to be a weak equivalence and the result follows; hence the first factorisation is proven. For the second factorisation choose p to be the weak equivalence in the original factorisation. This ensures that ι is a weak equivalence because the rest of the morphisms in the commutative square $\mathcal{L}(p)$, $i_{\mathfrak{g}}$, and \tilde{i} are weak equivalences. \square

One lift is given by the definition of the classes of morphism in the model structure. Therefore, it is only necessary to prove that all cofibrations have the left lifting property with respect to all acyclic fibrations.

Lemma 3.18. *Given a commutative diagram of the form*

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{a} \\ (f, \alpha) \downarrow & & \downarrow (\phi, \beta) \\ \mathfrak{h} & \longrightarrow & \mathfrak{b} \end{array},$$

where f is an acyclic cofibration and ϕ is a fibration, there exists a morphism (or lift) $(h, \gamma) : \mathfrak{h} \rightarrow \mathfrak{a}$ such that the diagram still commutes, i.e. acyclic cofibrations have the LLP with respect to all fibrations.

Proof. First, use Lemma 3.17 to factorise (f, α) as $(\tilde{p}, \beta) \circ (\tilde{i}, \gamma)$, where $(\tilde{p}, \beta) : \mathfrak{m} \rightarrow \mathfrak{h}$ is an acyclic fibration and $(\tilde{i}, \gamma) : \mathfrak{g} \rightarrow \mathfrak{m}$ is an acyclic cofibration. Note, both (\tilde{p}, β) and (\tilde{i}, γ) are acyclic by the 2 of 3 property. The factorisation is as in the proof of Lemma 3.17, meaning (\tilde{i}, γ) is obtained by a cobase change from a morphism obtained by applying \mathcal{L} to a surjective quasi-isomorphisms of unital edga: call it p .

Since (f, α) is a cofibration it has, by definition, the LLP with respect to all acyclic fibrations. In particular, the following commutative diagram exists:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{(\tilde{i}, \gamma)} & \mathfrak{m} \\ (f, \alpha) \downarrow & \nearrow (h, \eta) & \downarrow (\tilde{p}, \beta) \\ \mathfrak{h} & \xlongequal{\quad} & \mathfrak{h} \end{array},$$

and hence (f, α) is a retract of (\tilde{i}, γ) , as the following diagram shows:

$$\begin{array}{ccccc} \mathfrak{g} & \xlongequal{\quad} & \mathfrak{g} & \xlongequal{\quad} & \mathfrak{g} \\ (f, \alpha) \downarrow & & \downarrow (\tilde{i}, \gamma) & & \downarrow (f, \alpha) \\ \mathfrak{h} & \xrightarrow{(h, \eta)} & \mathfrak{m} & \xrightarrow{(\tilde{p}, \beta)} & \mathfrak{h} \end{array}.$$

Therefore, if (\tilde{i}, γ) has the LLP with respect to all fibrations the statement has been proven. Since (\tilde{i}, γ) is obtained by a cobase change of $\mathcal{L}(p)$, it suffices to show that $\mathcal{L}(p)$ has the LLP with respect to all fibrations. To this end, begin with the following diagram:

$$\begin{array}{ccc}
\mathcal{L}\mathcal{C}(\mathfrak{g}) & \longrightarrow & \mathfrak{x} \\
\mathcal{L}(p) \downarrow & & \downarrow (\varphi, \kappa) \\
\mathcal{L}(A) & \longrightarrow & \mathfrak{y}
\end{array}
,$$

where (φ, κ) is a fibration of curved Lie algebras. Applying the functor \mathcal{C} and extending the diagram gives:

$$\begin{array}{ccccc}
\mathcal{C}(\mathfrak{y}) & \longrightarrow & \mathcal{C}\mathcal{L}(A) & \xrightarrow{i_A} & A \\
\mathcal{C}(\varphi, \kappa) \downarrow & & & & \downarrow p \\
\mathcal{C}(\mathfrak{x}) & \longrightarrow & \mathcal{C}\mathcal{L}(\mathfrak{g}) & \xrightarrow{i_{\mathcal{C}(\mathfrak{g})}} & \mathcal{C}(\mathfrak{g})
\end{array}
.$$

Since p is an acyclic fibration and $\mathcal{C}(\varphi, \kappa)$ is a cofibration there exists a lift $H : \mathcal{C}(\mathfrak{x}) \rightarrow A$. Applying \mathcal{L} , composing with $i_{\mathfrak{x}}$ to obtain the morphism $\tilde{H} : \mathcal{L}(A) \rightarrow \mathfrak{x}$, and noticing it is the desired lift completes the proof. \square

With all the above, this now leads to the following result.

Theorem 3.19. *The category of curved Lie algebras with curved morphisms defines a model category with the model structure defined in Definition 3.11. Moreover, it is Quillen equivalent to the model category of unital commutative differential graded algebras.*

\square

4 Unbased rational homotopy theory

For the remainder of the paper it is assumed that the ground field is always the field of rational numbers, \mathbb{Q} . Within this section, results contained in [BG76] and [LM15] will be used alongside the Quillen equivalence proven in Theorem 3.19 of this paper to construct a disconnected rational homotopy theory using the category of pseudo-compact curved Lie algebras. The category of simplicial sets is denoted by \mathcal{S} . Recall that, the category \mathcal{S} possesses a well known model structure, which can be found for instance in [BG76] and [DS95], where the weak equivalences are weak homotopy equivalences. With this model structure all simplicial sets are cofibrant and the fibrant objects are the Kan complexes.

Definition 4.1. *A connected Kan complex is said to be*

- *nilpotent if its fundamental group is nilpotent and every other homotopy group is nilpotent as a π_1 module, for any choice of base vertex;*
- *rational if each of its homotopy groups are uniquely divisible.*

Given some category $X(= \hat{\mathcal{L}}_*, \mathcal{A}, \mathcal{S})$, let

- $\text{ho}(X)$ denote the homotopy category of X ,
- X^c denote the full subcategory of connected objects, and
- X^{dc} denoted the full subcategory of objects with finitely many connected components.

Therefore, with the above notation the objects of the category $\text{ho}(\mathcal{S}^c)$ are connected Kan complexes.

Definition 4.2. *An algebra is said to be connected if it is concentrated in non-negative degrees and equal to the ground field, \mathbb{Q} , in degree 0. Similarly, an algebra X is said to be homologically connected if $H^0(X) = \mathbb{Q}$ and $H^i(X) = 0$ for all $i < 0$. Extending this notion, one defines a homologically disconnected algebra to be one that is isomorphic to a finite product of homologically connected ones.*

Recall that every non-negatively graded homologically connected algebra admits a minimal model, c.f. [BG76, Section 7]. That is, every non-negatively graded homologically connected algebra is weakly equivalent to a minimal algebra. Moreover, the minimal algebra is unique up to isomorphism.

The adjective finite type will be understood to mean finite type over \mathbb{Q} and in the sense of [BG76]: that is

- a cofibrant homologically connected algebra is said to be of finite type if its minimal model has finitely many generators in each degree; and
- nilpotent connected Kan complex is said to be of finite type if its homology groups with coefficients in \mathbb{Q} are finite dimensional vector spaces over \mathbb{Q} .

The prefix $\text{f}\mathbb{N}\mathbb{Q}$ — applied to a category of simplicial sets will denote the full subcategory composed of rational, nilpotent objects of finite type over \mathbb{Q} , and the prefix $\text{f}\mathbb{Q}$ — applied to a category of algebras will denote the full subcategory of objects of finite type over \mathbb{Q} . For example, the category $\text{f}\mathbb{N}\mathbb{Q}\text{--ho}(\mathcal{S}^c)$ denotes the full subcategory of $\text{ho}(\mathcal{S})$ of rational, nilpotent Kan complexes of finite type.

[LM15, Theorem B] states that any homologically disconnected cdga is quasi-isomorphic to a finite product of homologically connected cdgas. Further, the homologically disconnected cdga is said to be of finite type if each connected cdga in the finite product is of finite type, see Proposition 4.4 in op. cit..

Let $\mathcal{A}_{\geq 0}$ denote the category of non-negatively graded cdgas. Recall, from [BG76, Theorem 9.4], there exists a pair of adjoint functors

$$F : \mathcal{A}_{\geq 0} \rightleftarrows \mathcal{S} : \Omega,$$

that induce an equivalence of the homotopy categories $\text{f}\mathbb{Q}\text{--ho}\mathcal{A}_{\geq 0}^c$ and $\text{f}\mathbb{N}\mathbb{Q}\text{--ho}\mathcal{S}^c$. This is the so-called Sullivan-de Rham equivalence. Here Ω is the de Rham functor [BG76, Section 2] and F is the functor given by $X \mapsto F(X, \mathbb{Q})$ where $F(X, \mathbb{Q})$ is the function space of [BG76, Section 5.1] (called the Bousfield-Kan functor). Note there is also an analogue of this result in the pointed case contained in op. cit.. Combining the equivalence of these homotopy categories with [LM15, Proposition 3.5]—where the authors extend the existence of minimal models to arbitrary homologically connected cdgas to prove the categories $\text{ho}\mathcal{A}_{\geq 0}^c$ and $\text{ho}\mathcal{A}^c$ are equivalent—it follows that the categories $\text{f}\mathbb{N}\mathbb{Q}\text{--ho}\mathcal{S}^c$ and $\text{f}\mathbb{Q}\text{--ho}\mathcal{A}^c$ are equivalent. Further, [LM15, Theorem C] states that the categories $\text{f}\mathbb{N}\mathbb{Q}\text{--ho}\mathcal{S}^{dc}$, $\text{f}\mathbb{Q}\text{--ho}\mathcal{A}_{\geq 0}^{dc}$, and $\text{f}\mathbb{Q}\text{--ho}\mathcal{A}^{dc}$ are equivalent.

It will be the aim of the rest of this section to describe a subcategory $\text{f}\mathbb{Q}\text{--ho}\hat{\mathcal{L}}_*^{dc}$ of $\text{ho}(\hat{\mathcal{L}}_*)$ that is also equivalent to the categories $\text{f}\mathbb{N}\mathbb{Q}\text{--ho}\mathcal{S}^{dc}$, $\text{f}\mathbb{Q}\text{--ho}\mathcal{A}_{\geq 0}^{dc}$, and $\text{f}\mathbb{Q}\text{--ho}\mathcal{A}^{dc}$. To this end, first note that any minimal algebra has a unique augmentation, thus being endowed with an augmentation means that this minimal algebra corresponds to a uncurved Lie algebra (a dgla) under the functor \mathcal{L} . This observation results in the corollary that for every homologically connected cdga, X , there exists a filtered quasi-isomorphism $\mathcal{L}(X) \rightarrow \mathcal{L}(M_X)$ since the functor \mathcal{L} preserves weak equivalences; here M_X is a minimal model for X . Thus, since a homologically disconnected algebra, A , is a finite product of homologically connected ones up to isomorphism, A is weakly equivalent to a finite product of minimal algebras, and hence $\mathcal{L}(A)$ is weakly equivalent to a finite coproduct of dglas of the form $\mathcal{L}(M_X)$ for some minimal algebra M_X . With this observation in mind the following definition is made.

Definition 4.3. *The category $\text{f}\mathbb{Q}\text{--ho}\hat{\mathcal{L}}_*^{dc}$ is the full subcategory of $\text{ho}(\hat{\mathcal{L}}_*)$ with objects consisting of finite coproducts of curved Lie algebras whose associated graded complexes have finite dimensional homology in each degree.*

This is an equivalent way to express full subcategory of $\text{ho}(\hat{\mathcal{L}}_*)$ with objects consisting of finite coproducts of curved Lie algebras of the form $\mathcal{L}(M)$, where M is a minimal cdga of finite type.

Clearly the category $\text{f}\mathbb{Q}\text{--ho}\hat{\mathcal{L}}_*^{dc}$ is equivalent to $\text{f}\mathbb{N}\mathbb{Q}\text{--ho}\mathcal{S}^{dc}$, $\text{f}\mathbb{Q}\text{--ho}\mathcal{A}_{\geq 0}^{dc}$, and $\text{f}\mathbb{Q}\text{--ho}\mathcal{A}^{dc}$. To describe the equivalence in a more specific manner, first recall the definition of the Maurer-Cartan simplicial set,

i.e. the functor $\mathrm{MC}_\bullet : \hat{\mathcal{L}}_* \rightarrow \mathcal{S}$ given in [HL09, Get09, LM15] for example. Note that the Sullivan-de Rham algebra of polynomial forms on the standard topological cosimplicial simplex must be considered as a homologically graded cdga so that the resulting object when tensored with a homologically curved Lie algebra is again a homologically curved Lie algebra.

Proposition 4.4. *The functors $\mathrm{MC}_\bullet : \hat{\mathcal{L}}_* \rightarrow \mathcal{S}$ and $\mathcal{L}\Omega : \mathcal{S} \rightarrow \hat{\mathcal{L}}_*$ form an adjoint pair.*

Proof. By definition, there exists an isomorphism of simplicial sets $\mathrm{MC}_\bullet(\mathfrak{g}) \cong FC(\mathfrak{g})$. Therefore, the functors are compositions in the following diagram:

$$\hat{\mathcal{L}}_* \begin{array}{c} \xleftarrow{\mathcal{C}} \\ \xrightarrow{\mathcal{L}} \end{array} \mathcal{A} \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{\Omega} \end{array} \mathcal{S}.$$

The functors \mathcal{L} and \mathcal{C} are adjoint by Theorem 3.19, and the functors F and Ω are adjoint by [BG76], therefore the result follows. \square

It further follows (from Proposition 2.6 and [BG76]) that the composite functors MC_\bullet and $\mathcal{L}\Omega$ induce adjoint functors upon the homotopy categories of $\hat{\mathcal{L}}_*$ and \mathcal{S} . Restricting to the categories $\mathrm{fN}\mathbb{Q}\text{-ho}\mathcal{S}^{dc}$ and $\mathrm{fQ}\text{-ho}\hat{\mathcal{L}}_*^{dc}$ these functors induce mutually inverse equivalences, as the following results show.

Proposition 4.5. *Given any connected non-negatively graded cdga, A , of finite type, the Lie algebra $\mathcal{L}(A)$ is weakly equivalent to an object belonging to the category $\mathrm{fQ}\text{-ho}\hat{\mathcal{L}}_*^c$.*

Proof. Since A is connected there exists some unique minimal model, M_A , that is also of finite type. Therefore, since A is quasi-isomorphic to M_A the Lie algebras $\mathcal{L}(A)$ and $\mathcal{L}(M_A)$ are weakly equivalent. \square

Proposition 4.6. *Given any curved Lie algebra $\mathfrak{g} \in \mathrm{fQ}\text{-ho}\hat{\mathcal{L}}_*^c$ the cdga $\mathcal{C}(\mathfrak{g})$ is quasi-isomorphic to a connected non-negatively graded cdga of finite type.*

Proof. The Lie algebra will be weakly equivalent to one of the form $\mathcal{L}(M)$ for some minimal algebra M of finite type, thus $\mathcal{C}\mathcal{L}(M)$ is quasi-isomorphic to M and thus of the right form. \square

Theorem 4.7. *The functors MC_\bullet and $\mathcal{L}\Omega$ determine mutually inverse equivalences between the categories $\mathrm{fQ}\text{-ho}\hat{\mathcal{L}}_*^{dc}$ and $\mathrm{fN}\mathbb{Q}\text{-ho}\mathcal{S}^{dc}$.*

Proof. First, note that the objects of the category $\mathrm{fQ}\text{-ho}\hat{\mathcal{L}}_*^{dc}$ are, up to equivalence, coproducts of objects of the category $\mathrm{fQ}\text{-ho}\hat{\mathcal{L}}_*^c$. Likewise the objects of the category $\mathrm{fQ}\text{-ho}\mathcal{A}^{dc}$ are, up to equivalence, products of objects of the category $\mathrm{fQ}\text{-ho}\mathcal{A}^c$. Hence to show there is an equivalence of $\mathrm{fQ}\text{-ho}\hat{\mathcal{L}}_*^{dc}$ and $\mathrm{fQ}\text{-ho}\mathcal{A}^{dc}$ it is sufficient to show there is an equivalence of $\mathrm{fQ}\text{-ho}\hat{\mathcal{L}}_*^c$ and $\mathrm{fQ}\text{-ho}\mathcal{A}^c$. By Propositions 4.5 and 4.6 the functors \mathcal{C} and \mathcal{L} restrict to an adjunction

$$\mathcal{C} : \mathrm{fQ}\text{-ho}\hat{\mathcal{L}}_*^c \rightleftarrows \mathrm{fQ}\text{-ho}\mathcal{A}^c : \mathcal{L}.$$

Since Theorem 3.19 shows that the functors \mathcal{C} and \mathcal{L} form a Quillen pair it is evident that their restrictions form mutually inverse equivalences of the categories $\mathrm{fQ}\text{-ho}\hat{\mathcal{L}}_*^c$ and $\mathrm{fQ}\text{-ho}\mathcal{A}^c$. Whence the categories $\mathrm{fQ}\text{-ho}\hat{\mathcal{L}}_*^{dc}$ and $\mathrm{fQ}\text{-ho}\mathcal{A}^{dc}$ are equivalent. Combining this with [LM15, Theorem C] there exists equivalences of the categories

$$\mathrm{fQ}\text{-ho}\hat{\mathcal{L}}_*^{dc} \sim \mathrm{fQ}\text{-ho}\mathcal{A}^{dc} \sim \mathrm{fQ}\text{-ho}\mathcal{A}_{\geq 0}^{dc} \sim \mathrm{fN}\mathbb{Q}\text{-ho}\mathcal{S}^{dc},$$

and the proof is complete. \square

Since equivalences of categories preserve colimits and limits, it should be noted that an analogue of Theorem 1.7 in [LM15] and its corollary can be explained here as a result of Theorem 4.7. More precisely, analogues of the results can be shown here as simple corollaries if one allows oneself to work up to homotopy.

Corollary 4.8. *Given $\coprod_{i \in I} \mathfrak{g}_i \in \mathrm{fQ}\text{-ho}\hat{\mathcal{L}}_*^{dc}$, the simplicial set $\mathrm{MC}_\bullet(\coprod_{i \in I} \mathfrak{g}_i)$ is weakly equivalent to the disjoint union $\bigcup_{i \in I} \mathrm{MC}_\bullet(\mathfrak{g}_i)$.*

Let $\mathcal{MC}(-) := \pi_0(\mathcal{MC}_\bullet(-))$ denote the Maurer-Cartan moduli set, i.e. the set of Maurer-Cartan elements up to homotopy. This construction can be found for example in [HL09, Get09, LM15]. An alternate construction in the general case of pronilpotent dglas given in [SS12] is that the Maurer-Cartan moduli set can be described as the Maurer-Cartan set up to gauge equivalence. For more details regarding gauge equivalence and a proof of this statement, see [CL10].

Corollary 4.9. *Given $\coprod_{i \in I} \mathfrak{g}_i \in \mathbf{fQ}\text{-ho}\hat{\mathcal{L}}_*^{dc}$, there exists an isomorphism of sets*

$$\mathcal{MC}\left(\prod_{i \in I} \mathfrak{g}_i\right) \cong \bigcup_{i \in I} \mathcal{MC}(\mathfrak{g}_i).$$

Theorem 4.7 also has an application to mapping spaces, but before stating this application some important related facts must be recalled. First, recall that given a pseudo-compact curved Lie algebra, $(\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}})$, and a unital cdga, A , (both homologically graded) their completed tensor product possesses a well defined pseudo-compact curved Lie algebra structure: the curvature is given by $\omega_{\mathfrak{g}} \hat{\otimes} 1$, the differential is defined on elementary tensors by $d(x \hat{\otimes} a) = d_{\mathfrak{g}}x \hat{\otimes} a + (-1)^{|x|}x \hat{\otimes} d_A a$, and the bracket is defined on elementary tensors by $[x \hat{\otimes} a, y \hat{\otimes} b] = [x, y] \hat{\otimes} (-1)^{|y||a|}ab$. Given such a tensor product, the MC elements can be studied in the usual manner as solutions to the MC equation

$$\omega_{\mathfrak{g}} \hat{\otimes} 1 + d\xi + \frac{1}{2}[\xi, \xi].$$

The MC elements of such a tensor product correspond to morphisms of cdgas. More precisely, given some degree minus one element of $\mathfrak{g} \hat{\otimes} A$, it corresponds precisely to a continuous linear morphism $k \rightarrow (\Sigma^{-1}\mathfrak{g}) \hat{\otimes} A$ of degree zero. This continuous linear morphism, in turn, defines (and is defined by) a linear morphism $(\Sigma^{-1}\mathfrak{g})^* \rightarrow A$ which extends uniquely to a morphism of graded commutative algebras $\mathcal{C}(\mathfrak{g}) \rightarrow A$. The condition that this morphism is in fact one of cdga is precisely the one that the original element is a MC element. Therefore, viewing $\mathcal{MC}(\mathfrak{g} \hat{\otimes} A) = \mathcal{MC}(\mathfrak{g}, A)$ as a bifunctorial construction the following proposition follows.

Proposition 4.10. *Given a pseudo-compact curved Lie algebra, $(\mathfrak{g}, d_{\mathfrak{g}}, \omega_{\mathfrak{g}})$, and a cdga, A , the two functors $\mathcal{MC}(\mathfrak{g}, A)$ and $\text{Hom}_{\mathcal{A}}(\mathcal{C}(\mathfrak{g}), A)$ are naturally isomorphic.*

This result extends to the level of homotopy; it is necessary to first, however, recall the definition of a homotopy of MC elements.

Definition 4.11. *Let $k[z, dz]$ be the free unital cdga on generators z and dz of degrees 0 and 1 respectively, subject to the condition $d(z) = dz$.*

Given some unital cdga A , let $A[z, dz]$ denote the unital cdga given by the tensor $A \otimes k[z, dz]$. Further, denote the quotient morphisms by setting $z = 0, 1$ by $|_0, |_1 : A[z, dz] \rightarrow A$.

Definition 4.12. *Given a pseudo-compact curved Lie algebra, \mathfrak{g} , and a unital cdga, A , two elements $\xi, \eta \in \mathcal{MC}(\mathfrak{g}, A)$ are said to be homotopic if there exists $h \in \mathcal{MC}(\mathfrak{g}, A[z, dz])$ such that $h|_0 = \xi$ and $h|_1 = \eta$.*

As Proposition 4.10 shows a homotopy of MC elements is nothing more than a Sullivan homotopy, i.e. a right homotopy with path object $A[z, dz]$. Therefore, two elements of $\mathcal{MC}(\mathfrak{g}, A)$ belong to the same class if, and only if, the two corresponding morphisms $\mathcal{C}(\mathfrak{g}) \rightarrow A$ belong to the same homotopy class.

Corollary 4.13. *Given $X, Y \in \mathbf{fNQ}\text{-ho}\mathcal{S}^{dc}$, then*

$$\text{Hom}_{\mathbf{fNQ}\text{-ho}\mathcal{S}^{dc}}(X, Y) \cong \mathcal{MC}(\mathcal{L}\Omega(Y) \otimes \Omega(X)).$$

Proof. First, there is clearly an isomorphism

$$\text{Hom}_{\mathbf{fNQ}\text{-ho}\mathcal{S}^{dc}}(X, Y) \cong \text{Hom}_{\mathbf{fQ}\text{-ho}\mathcal{A}^{dc}}(\Omega(Y), \Omega(X)).$$

It therefore suffices to show that there exists an isomorphism of $\mathcal{MC}(\mathcal{L}\Omega(Y) \otimes \Omega(X))$ and homotopy classes of morphisms $\Omega(Y) \rightarrow \Omega(X)$ which is contained within the discussion above. \square

Remark 4.14. In [Laz13, Theorem 8.1] an explicit model for every connected component of the mapping space between two connected rational, nilpotent CW complexes of finite type, X and Y , is given; it is further assumed that either X is a finite CW complex or Y has a finite Postnikov tower, because this ensures the spaces of maps between X and Y are homotopically equivalent to finite type complexes. Whereas the result Corollary 4.13 gives a model for the whole mapping space. Therefore, in the case when the two spaces, X and Y , are both sufficiently nice (i.e. are both composed of finitely many connected components each rational, nilpotent, and of finite type with either X being a finite CW complex or Y having a finite Postnikov tower and such that the space of maps has finitely many connected components), the results [Laz13, Theorem 8.1] and Corollary 4.13 could be combined to construct a model for the mapping space as a coproduct of finitely many MC moduli spaces. The material developed in this paper, however, does not allow the extension to the case where at least one of the spaces, X or Y , fails to meet the aforementioned constraints, or to the case when the space of maps has infinitely many connected components, and this can happen in some seemingly straightforward cases; the space of maps between 2-dimensional spheres, for example, has infinitely many connected components.

References

- [Bar68] Michael Barr. Harrison homology, Hochschild homology and triples. *J. Algebra*, 8:314–323, 1968.
- [BG76] A. K. Bousfield and V. K. A. M. Gugenheim. On PL de Rham theory and rational homotopy type. *Mem. Amer. Math. Soc.*, 8(179):ix+94, 1976.
- [BL05] J. Block and A. Lazarev. André-Quillen cohomology and rational homotopy of function spaces. *Adv. Math.*, 193(1):18–39, 2005.
- [Bra12] Christopher Braun. Operads and moduli spaces. *arXiv:1209.1088*, 2012.
- [CL10] J. Chuang and A. Lazarev. Feynman diagrams and minimal models for operadic algebras. *J. Lond. Math. Soc. (2)*, 81(2):317–337, 2010.
- [CL11] Joseph Chuang and Andrey Lazarev. L -infinity maps and twistings. *Homology Homotopy Appl.*, 13(2):175–195, 2011.
- [CLM14] Joseph Chuang, Andrey Lazarev, and Wajid Mannan. Cocommutative coalgebras: homotopy theory and Koszul duality. *arXiv:1403.0774*, 2014.
- [DS95] William G. Dwyer and Jan Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.
- [Gar62] Pierre Garbiel. Des catégories abéliennes. *Bull. Soc. Math. France*, 90:323–448, 1962.
- [Get09] Ezra Getzler. Lie theory for nilpotent L_∞ -algebras. *Ann. of Math. (2)*, 170(1):271–301, 2009.
- [Har62] D. K. Harrison. Commutative algebras and cohomology. *Trans. Amer. Math. Soc.*, 104:191–204, 1962.
- [Hin97] Vladimir Hinich. Homological algebra of homotopy algebras. *Comm. Algebra*, 25(10):3291–3323, 1997.
- [Hin01] Vladimir Hinich. DG coalgebras as formal stacks. *J. Pure Appl. Algebra*, 162(2-3):209–250, 2001.
- [HL09] Alastair Hamilton and Andrey Lazarev. Cohomology theories for homotopy algebras and noncommutative geometry. *Algebr. Geom. Topol.*, 9(3):1503–1583, 2009.
- [KY11] Bernhard Keller and Dong Yang. Derived equivalences from mutations of quivers with potential. *Adv. Math.*, 226(3):2118–2168, 2011.

- [Laz13] Andrey Lazarev. Maurer-Cartan moduli and models for function spaces. *Adv. Math.*, 235:296–320, 2013.
- [LM15] Andrey Lazarev and Martin Markl. Disconnected rational homotopy theory. *Adv. Math.*, 283:303–361, 2015.
- [ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [Pos11] Leonid Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Mem. Amer. Math. Soc.*, 212(996):vi+133, 2011.
- [Qui69] Daniel Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.
- [SS12] Michael Schlessinger and James Stasheff. Deformation theory and rational homotopy type. *arXiv:1211.1647*, 2012.
- [Sul77] Dennis Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (47):269–331 (1978), 1977.
- [VdB15] Michel Van den Bergh. Calabi-Yau algebras and superpotentials. *Selecta Math. (N.S.)*, 21(2):555–603, 2015.
- [Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.